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Exact solutions to a nonlinear reaction–diffusion equation and hyperelliptic integrals inversion

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Abstract. An approach is proposed to obtain some exact explicit stationary solutions in terms of elliptic functions to a nonlinear reaction–diffusion equation. The method is based on the reduction of the hyperelliptic integral to the elliptic one and its inversion via the Weierstrass and Jacobi elliptic functions. The solutions for both polynomial reaction and diffusion functions include bounded periodic and localized (in space) functions. Such solutions seem to be the best candidates to describe periodic nanostructures observed in experiments on formation of thin films by means of molecular epitaxy (the so-called 'quantum wires'). Generalization of the approach is discussed for reaction and diffusion functions distinctive from polynomials. In particular, explicit stationary solutions are found in terms of elliptic functions for arbitrary diffusion and relevant reaction terms.

1. Introduction

In this paper we propose an approach to find some exact stationary quasi-periodic solutions to a nonlinear diffusion equation with a reaction term $A(u)$:

$$u'_t = (D(u)u'_x)'_x + A(u) \tag{1}$$

where both $D(u)$ and $A(u)$ are known functions of $u(x, t)$.

Depending on a particular form of these functions, equation (1) appears in population genetics, combustion theory, continuum physics, selforganization phenomena, interphase interactions physics, etc [1–11]. The most common versions of the nonlinear equation considered are known as the KPP or Fisher equation, occurring in population dynamics and combustion theory. Some of the simplest exact solutions were found [3] by means of phase plane analysis. Numerical analysis has been widely used to study the modified Selkov model for cubic chemical reaction in hyperbolic or parabolic diffusion limit, see e.g. [4], while in [5] the lattice Boltzmann equation was used to yield a parabolic reaction–diffusion model. However, complete description of the possible set of exact solutions seems not to have been achieved, even for an ODE version of a nonlinear problem, having both arbitrary reaction and diffusion terms. An important problem arises in the derivation of exact solutions (and, in particular, of exact explicit solutions for the corresponding stationary equation) widely used as test points in numerical simulations.

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The problem under consideration has also been studied by means of group theory methods. The complete group of Lie point symmetries for equation (1) with various reaction and diffusion functions can be found in [8]. Corresponding invariant solutions for blow-up process modelling in heat conduction are widely used, see [7]. Nonclassical symmetries of (1) were considered in [6], and the corresponding new invariant solutions were obtained for exponential and power diffusion terms. The description of particular nonclassical symmetries of (1) with arbitrary diffusion function was given in [6], which led to some exact periodic invariant solutions.

We are aiming to obtain some new exact periodic and localized solutions by means of a different approach. Most of the solutions to equation (1) obtained for either exponential or power law are assumed to be valid for diffusion and reaction functions. Surprisingly, in other cases symmetry methods almost always provide only trivial group solutions. The connection between symmetry methods and *ad hoc* methods of derivation of exact solutions is still not clear, which makes the latter methods important for the understanding of the different effects governed by (1). On the other hand, the methods we use are based on the classical results of the theory of hyperelliptic integrals and their reduction to elliptic ones, which seems to be useful in many applications.

The paper is organized as follows. Sections 2 and 3 are devoted to description of the method for obtaining exact solutions to equation (1) with polynomial $D(u)$ and $A(u)$ in terms of elliptic functions, that is based on reductions of hyperelliptic integrals. In section 4 we obtain the corresponding bounded periodic and localized solutions. Section 5 is devoted to the numerical analysis of physical parameters providing the bounded solutions. Generalization of the approach for both $A(u)$ and $D(u)$ distinctive from polynomials is given in section 6.

Equation (1), containing both *polynomial* reaction and diffusion terms, seems to be the most important for physical applications. Here, we briefly discuss its derivation in the relatively new mathematical model [12,13] of description of the adsorbate growth kinetics on a surface by means of molecular beam epitaxy, which is widely used in microelectronics. Such processes are usually described in terms of phenomenological approaches to the thermodynamics of irreversible processes, lattice gas models and their modifications (see, e.g. [12–14]). Within the framework of phenomenology the question arises of how to determine a connection between the growth kinetics and the description of adsorption, desorption and diffusion elementary processes. On the other hand, the use of lattice gas models does not necessarily lead to appropriate formulation of a single kinetic equation, in which all elementary processes are taken into account, and which is useful for description of adsorbate growth at all stages. A starting point for finding a relationship between phenomenological and microscopic theories of adsorbate metastability seems to be given by the so-called generalized kinetic Brunauer–Emmet–Teller (BET) model for multilayer adsorbate growth from a one-component gas under isothermal conditions [12–14]. This model is intermediate between the master equation method [15] and the BET model [16], and deals with the microscopic approach based on time-dependent lattice gas models. An advantage of this model is that it makes it possible to consider the adsorbate growth kinetics taking into account the influence of different microscopic processes, adsorption regimes, inhomogeneities in the substrate and molecular beam, without using phenomenological parameters and additional assumptions on a growth mechanism. The adequacy of this model was confirmed both by its correct limit transitions to the classical results of the adsorption theory and the experimental results [17, 18].

In the case of the monolayer growth dynamics under natural constraints for the BET model, the continuum one-dimensional limit of the master equation for the probability $\theta(\vec{r}, t)$ of finding a particle in a surface point $\vec{r} = (x, y)$ at time t takes the form

$$\frac{\partial\theta(\vec{r}, t)}{\partial t} = \nabla_{\vec{r}}(D(\theta)\nabla_{\vec{r}}\theta) + A(\theta) \qquad (2)$$

$$A(\theta) = b(1-\theta) - \theta \exp(-\phi_A \theta) \qquad D(\theta) = [1 - \phi_D \theta(1-\theta)] \exp(-\phi_D \theta) \tag{3}$$

where b, ϕ_A and ϕ_D are given physical parameters of the model.

It is easy to see that in order to obtain a solution to (2) which does not depend on space and time variables, one has to solve the following transcendental equation:

$$A(\theta) = b(1-\theta) - \theta \mathrm{e}^{-\phi_A \theta} = 0. \tag{4}$$

For values of b belonging to the open interval $b_{\min} < b < b_{\max}$ (where $b_{\min} = (\theta_2^A/\theta_1^A) \times \exp(-\phi_A \theta_2^A)$; $b_{\max} = (\theta_1^A/\theta_2^A) \exp(-\phi_A \theta_1^A)$; and $\theta_{1,2}^A = (1 \mp \frac{1}{2}\sqrt{1 - 4/\phi_A}))$ there are at most three solutions ($\theta_1 \leqslant \theta_2 \leqslant \theta_3$) to (4). A root θ_2 is thermodynamically unstable and, in the case of various roots of (4), it describes the coexistence of a gas phase with chemical potential proportional to $\log b$ and two adsorbate phases with constant coverage levels θ_1 (dilute phase) and θ_3 (dense phase), see [13]. Hence, it is of interest for physics to study equation (2) for those θ that are in a small vicinity of roots of equation (4). Introducing $\theta = \theta_0 + u$, where θ_0 is a root of (4), and writing exponential terms in (3) as power series with respect to u and omitting higher-order terms, one can obtain the one-dimensional stationary version of (2) in the form

$$(P_s(u)u')' + Q_n(u) = 0 \tag{5}$$

where

$$\begin{aligned} P_s(u) &= 1 + p_1 u + p_2 u^2 + \cdots + p_s u^s \\ Q_n(u) &= u(q_0 + q_1 u + \cdots + q_{n-1} u^{n-1}) \end{aligned} \tag{6}$$

and coefficients p_i, q_i are given functions of physical parameters. These functions are evidently not zero, which makes the problem of derivation of exact solutions to (5) with arbitrary polynomials P_s and Q_n very important.

Equation (5) also arises in many other applications. For example, it represents the stationary version of the generalized Fisher equation used in population genetics. For this reason we first consider (5) for arbitrary values of u. After the solutions to (5) are obtained, they are analysed (in sections 4 and 5) for $0 < u < 1$.

2. Analysis of the stationary equation

The stationary version of equation (1) in the form

$$(D(u)u')' + A(u) = 0 \tag{7}$$

for *arbitrary* functions $A(u)$ and $D(u)$ can be reduced by means of the transformation [26]

$$v(u) = (u')^2$$

to the following linear equation:

$$v' + f(u)v = g(u)$$

where

$$f(u) = (\log D^2(u))'_u \qquad g(u) = -2A(u)/D(u).$$

Solving it, we obtain the following implicit solution to equation (7):

$$x = c_2 + \int \frac{D(u)}{\sqrt{c_1 - 2\int A(u)D(u)\,\mathrm{d}u}}\,\mathrm{d}u \tag{8}$$

where c_1 and c_2 are arbitrary integration constants.

The problem of derivation of exact explicit solutions to equation (7) with arbitrary functions $A(u)$ and $D(u)$ is reduced, thereby, to the inversion of the integral in (8).

Now assume that both diffusion and adsorption functions are polynomials, $D(u) = P_s(u)$, $A(u) = Q_n(u)$, of the form (6) with coefficients p_i, q_j to be specified by a problem under consideration. Let $p_0 = 1$ and $Q_n(0) = 0$ be assumed for simplicity: this is quite typical for various physical problems; however, an analysis can be done in a similar way for arbitrary polynomials $P_s(u)$, $Q_n(u)$ of orders s and n, respectively.

Due to the assumption, (8) leads to the following solution to equation (7):

$$x = c_2 + \int \frac{P_s(u)}{\sqrt{c_1 - 2\int Q_n(u)P_s(u)\,\mathrm{d}u}}\,\mathrm{d}u \equiv c_2 + \int \frac{P_s(u)}{\sqrt{R_{s+n+1}(u)}}\,\mathrm{d}u. \tag{9}$$

The integral in (9) is the Abelian one. The inversion problem for Abelian integrals of the first kind, i.e. the Jacobi inversion problem [19], is solved in terms of Abelian functions, which are the single-valued, analytic, k-dimensional, and $2k$-periodic functions. Explicit formulae for the solutions can be obtained in terms of multi-dimensional Riemann θ-functions. In particular, for $k = 1$, elliptic functions will constitute the appropriate limits. The corresponding Jacobi inversion problem coincides with the inversion problem for the elliptic integral of the first kind, and relevant explicit formulae for solutions represent the well known relationship between the Jacobi and Weierstrass elliptic functions and the Jacobi θ-functions, [27].

An elliptic integral of the first kind appears in (9) if $s = 0$ and $n = 2, 3$. Calculating its inverse, a solution to equation (7) in terms of elliptic functions can be obtained, when a constant diffusion and a polynomial adsorption of fourth order are given. The solution depends on two arbitrary constants c_1 and c_2.

For $s > 0$ and $n > 2$ the hyperelliptic integral in (9) cannot have a single-valued inverse function in the general case. However, in some particular cases a hyperelliptic integral of the first kind is reduced to an elliptic integral of the first kind, which is invertible in terms of elliptic functions introduced by Jacobi and Weierstrass. For such reduction to be valid one has to impose corresponding constraints for numbers s, n and for coefficients of polynomials $P_s(u)$ and $Q_n(u)$.

2.1. Simplest reduction

Factorization of the polynomial $R_{s+n+1}(u)$ seems to be the simplest case of the reduction of (9). Namely, it results in an elliptic integral of the first kind, if the following relationship is valid:

$$R_{s+n+1}(u) = [P_s(u)]^2 \tilde{R}_{3;4}(u) \tag{10}$$

where $\tilde{R}_{3;4}(u)$ is a polynomial of third or fourth order. It is easy to see from (10) that the factorization is possible only if $n - s - 2 = 0$ or $n - s - 3 = 0$.

Let us assume $n = s + 3$ with arbitrary s (which corresponds to the polynomial $\tilde{R}_4(u)$ in (10)). Collecting coefficients with the same power of u in the left-hand side of (10) and making them equal to the corresponding terms in the right-hand side, one can reduce (10) to a linear system of algebraic equations, in which the coefficients of $\tilde{R}_4(u)$ are variables. If

$b_{2s} \equiv p_s^2 \neq 0$, then the basis minor Ω of the resulting system has the form

$$\Omega = \begin{pmatrix} b_{2s} & b_{2s-1} & b_{2s-2} & b_{2s-3} & b_{2s-4} \\ 0 & b_{2s} & b_{2s-1} & b_{2s-2} & b_{2s-3} \\ 0 & 0 & b_{2s} & b_{2s-1} & b_{2s-2} \\ 0 & 0 & 0 & b_{2s} & b_{2s-1} \\ 0 & 0 & 0 & 0 & b_{2s} \end{pmatrix}$$
$$b_k = \sum_{i,j=0}^{s} \delta_{i+j}^k p_i p_j \qquad k = 0, \ldots, 2s \qquad \text{and} \qquad b_k = 0 \quad k < 0 \tag{11}$$

where p_i are from (6) and δ_i^k is the Kronecker's symbol. Let us also denote

$$\Lambda_i = \begin{pmatrix} & & & r_{2s} \\ & \Omega & & \vdots \\ & & & r_{2s+4} \\ b_i & \ldots & b_{i-4} & r_i \end{pmatrix} \qquad i = 0, \ldots, 2s-1$$
$$r_0 = c_1 \qquad r_k = -\frac{2}{k} \sum_{(i,j)=(0,1)}^{(s,n)} \delta_{i+j}^{k-1} p_i q_{j-1} \qquad k = 1, \ldots, s+n+1 \tag{12}$$

where q_j are from (6). Note that $r_1 = 0$; however, it does not provide any simplification of the solution below. Using standard linear algebra, one can conclude that, if and only if the following condition is valid:

$$\det \Lambda_i = 0 \qquad i = 0, \ldots, 2s-1 \tag{13}$$

then the famous Cramer formulae yield the following unique solution $(\xi_0, \ldots, \xi_4)$ (i.e. the coefficients of the polynomial $\tilde{R}_4(u)$) to the algebraic system under consideration:

$$\xi_i = \frac{1}{b_{2s}^5} \sum_{j=1}^{5} r_{2s+j-1} \Omega_{j,i+1} \qquad i = 0, \ldots, 4 \tag{14}$$

where $\Omega_{j,i+1}$ is the minor of the matrix element $(\Omega)_{j,i+1}$. Thus, we are ready to formulate the following proposition.

Proposition 1. *If for arbitrary s, $n = s + 3$ and $p_s \neq 0$ the conditions (13) on the coefficients of equation (5) are valid, then this equation has the following solution:*

$$u(x) = \alpha - \frac{2a_1}{2\wp(x + c_2; g_2, g_3) - a_2} \tag{15}$$

where α is a root of the polynomial $\tilde{R}_4(u) \equiv \xi_0 + \xi_1 u + \cdots + \xi_4 u^4$, ξ_i are given in (14), and the constants

$$a_1 = \tfrac{1}{4} P'(\alpha) \qquad a_2 = \tfrac{1}{12} P''(\alpha) \qquad a_3 = \tfrac{1}{24} P'''(\alpha) \qquad a_4 = \tfrac{1}{24} P^{(4)}(\alpha)$$

define the invariants

$$g_2 = 3a_2^2 - 4a_1 a_3 \qquad g_3 = 2a_1 a_2 a_3 - a_2^3 - a_1^2 a_4.$$

Proof. If (13) holds, then the factorization (10) takes place with $\tilde{R}_4(u)$ as written above. Hence, the integral in (9) is elliptic of the first kind, and formula (15) is the result of its inversion. That completes the proof. □

Repeating the scheme described above with obvious changes in the case $n = s + 2$ (i.e. for the cubic $\tilde{R}_3(u)$ in (10)), one obtains the following result.

Proposition 2. *If for any arbitrary s, $n = s + 2$ and $p_s \neq 0$ in $P_s(u)$ the following conditions on the coefficients of the equation (5) are valid:*

$$\det \tilde{\Lambda}_i = 0 \qquad i = 0, \ldots, 2s - 1 \tag{16}$$

then equation (5) has the following solution:

$$u(x) = \frac{4}{\zeta_3}\left(\wp(x + c_2; g_2, g_3) - \frac{\zeta_2}{12}\right) \tag{17}$$

where

$$\zeta_i = \frac{1}{b_{2s}^4}\sum_{j=1}^{4} r_{2s+j-1}\tilde{\Omega}_{j,i+1} \qquad i = 0, \ldots, 3$$

$$g_2 = \tfrac{1}{12}(\zeta_2^2 - 3\zeta_1\zeta_3) \qquad g_3 = \tfrac{1}{432}(9\zeta_1\zeta_2\zeta_3 - 27\zeta_0\zeta_3^2 - 2\zeta_2^3).$$

Here matrices $\tilde{\Omega}$ and $\tilde{\Lambda}_i$ are formed from matrices Ω and Λ_i by omitting the fifth column and the fifth row, respectively.

2.2. Reductions in the case of linear diffusion and cubic or quartic adsorption

Factorization of the form (10) is not the only way resulting in successful inversion of the integral in (9). Methods of Riemann surfaces geometry allow to obtain some other exact solutions.

The most developed case of the hyperelliptic integral reduction occurs when $g = 2$, where g is genus of the Riemann surface corresponding to the integral in (9), that is, the case $s = 1$ and $n = 3$ or 4. The problem of such reduction attracted much attention [24, 25] in the theory of nonlinear equations integrable by the inverse scattering transform method due to a possibility to reduce the problem of periodic solutions of such equations to the Jacobi inversion problem [20]. Necessary and sufficient conditions are known [24] for the reduction of an arbitrary hyperelliptic integral of the first kind to the elliptic one formulated in terms of the 2×2 matrix $\{B_{ij}\}$ of the hyperelliptic integral periods on the corresponding Riemann surface:

$$k_1 + k_2 B_{11} + k_3 B_{12} + k_4 B_{22} + k_5(B_{11}B_{22} - B_{12}^2) = 0 \tag{18}$$

for some integer k_i. The problem of formulating criterion (18) in terms of the polynomial coefficients in a hyperelliptic integral (i.e. explicitly in terms of the coefficients of the initial differential equation) is far from being solved. However, besides the reduction by means of factorization of $R_{s+n+1}(u)$ in the case $s = 1, n = 3, 4$ one can use the reduction examples, which were found in the case $g = 2$ by Jacobi, Hermite, Burnside and Bolza.

The Jacobi formula [21,24] describes all possible reductions for k_i satisfying the following relation:

$$k_3^2 + 4(k_1k_5 - k_2k_4) = 4$$

and has the form

$$\int \frac{(1 + cy)\,\mathrm{d}y}{\sqrt{y(y-1)(y-a)(y-b)(y-ab)}} = \frac{1}{2}\left[\left(\frac{1}{\sqrt{ab}} + c\right)J_+ - \left(\frac{1}{\sqrt{ab}} - c\right)J_-\right]$$

where

$$J_\pm = \int \frac{\mathrm{d}z}{\sqrt{z(1-z)(1-(c_\pm)^2 z)}}$$

$$z = \frac{(1-a)(1-b)y}{(y-a)(y-b)} \qquad (c_\pm)^2 = -\frac{(\sqrt{a} \mp \sqrt{b})^2}{(1-a)(1-b)}.$$

The Burnside reduction [22, 24] is given by the formulae

$$\int \frac{(1+2c)(y+1/2) - c(2+c)}{w}\,\mathrm{d}y = \int \frac{1}{\sqrt{\xi(\xi-1)(\xi-y_0)}}\,\mathrm{d}\xi \tag{19a}$$

$$\int \frac{(y+c+\frac{1}{2})}{w}\mathrm{d}y = \int \frac{1}{\sqrt{(\zeta-\zeta_1)(\zeta-\zeta_2)(\zeta-\zeta_3)}}\,\mathrm{d}\zeta \tag{19b}$$

$$w^2 = (y^2 - \tfrac{1}{4})\{(y+\tfrac{1}{2})(y+c^2+2c+\tfrac{1}{2})^2 - y_0[(1+2c)(y+\tfrac{1}{2})+c^2]^2\} \tag{19c}$$

$$\xi = \frac{(y+1/2)(y+1/2+2c+c^2)^2}{[(1+2c)(y+1/2)+c^2]^2} \tag{19d}$$

$$\zeta = \frac{(y+1/2)^3 + c^2(y+1/2) + 2c - \xi_0[(y+1/2)(1+2c)+c^2]}{y^2 - 1/4} \tag{19e}$$

where c, y_0 and ξ_0 are arbitrary constants (in particular, for $c = -\frac{1}{2}$ formulae (19) represent the Hermite reduction [24]).

For $s = 1, n = 3$ the use of these reductions yields the elliptic integrals in the implicit solution (9). Inverting them, we obtain the following propositions.

Proposition 3 (Jacobi reduction). *Let $z_1, \ldots, z_5$ be roots of the polynomial $R_5(u)$ from (9) (for simplicity they are supposed to be different), and the following conditions be valid:*

$$\begin{aligned} &(z_2 - z_1)(z_5 - z_1) = (z_3 - z_1)(z_4 - z_1) \\ &p_0 + p_1 z_1 = \pm p_1\sqrt{(z_3 - z_1)(z_4 - z_1)}. \end{aligned} \tag{20}$$

Then the equation $(P_1(u)u')' + Q_3(u) = 0$ has a family of solutions

$$\begin{aligned} &u(x) = (z_2 - z_1)X(v_\pm(x)) + z_1 \\ &v_\pm(x) = \frac{4}{c_\pm^2}\wp\left[\frac{(z_2-z_1)\sqrt{r_5(z_5-z_1)}}{p_0+p_1z_1}(x+c_2);\, g_{2,\pm}, g_{3,\pm}\right] + \frac{1+c_\pm^2}{3c_\pm^2} \end{aligned} \tag{21}$$

where

$$c_\pm^2 = -\frac{\sqrt{z_2 - z_1}\left(\sqrt{z_3 - z_1} \mp \sqrt{z_4 - z_1}\right)^2}{(z_2 - z_3)(z_2 - z_4)}$$

$$g_{2,\pm} = \tfrac{1}{12}[(1+c_\pm^2)^2 - 2c_\pm^2] \qquad g_{3,\pm} = \tfrac{1}{216}(1+c_\pm^2)[(1+c_\pm^2)^2 - 3c_\pm^2]$$

and $X(\mu)$, being the function of its unknown μ, is defined as a root of an algebraic equation

$$\mu X^2 - \left(\mu\frac{z_3+z_4-2z_1}{z_2-z_1} + \frac{(z_2-z_3)(z_2-z_4)}{(z_2-z_1)^2}\right)X + \mu\frac{z_5-z_1}{z_2-z_1} = 0.$$

Recall that r_5 is defined in (12); indices '+' and '−' in the above formulae correspond to '+' and '−' signs in (20), respectively.

Proposition 4 (Burnside reduction). *(i) Let the coefficients p_i, q_i be such that the integral in the implicit solution (9) is reducible to the integral in the left-hand side of (19a), i.e.*

$$\frac{P_1(y)}{\sqrt{R_5(y)}} = \frac{(1+2c)(y+1/2) - c(2+c)}{w} \qquad \forall y \in \mathbb{R}$$

where w is from (19c). Then the equation $(P_1(u)u')' + Q_3(u) = 0$ has the following solution in terms of the Weierstrass elliptic function:

$$\begin{aligned} &u(x) = Y(v(x)) \\ &v(x) = 4\wp\left[\sqrt{r_5}(x+c_2);\, g_2, g_3\right] + \tfrac{1}{3}(1+y_0) \end{aligned} \tag{22}$$

where

$$g_2 = \tfrac{1}{12}[(1+y_0)^2 - 2y_0] \qquad g_3 = \tfrac{1}{216}(1+y_0)[(1+y_0)^2 - 3y_0]$$

with y_0 *and* c *being arbitrary constants, and the function* $Y(\mu)$ *defined as a root of the cubic equation* $Y^3 + C_2(\mu)Y^2 + C_1(\mu)Y + C_0(\mu) = 0$ *with the following coefficients:*

$$\begin{aligned}
C_0(\mu) &= \tfrac{1}{8}(1-2\mu) + (1-\mu)c + (\tfrac{5}{2} - 2\mu)c^2 + 2(1-\mu)c^3 + (\tfrac{1}{2} - \mu)c^4 \\
C_1(\mu) &= \tfrac{3}{4} - \mu + 4(1-\mu)c + 6(1-\mu)c^2 + 4(1-\mu)c^3 + c^4 \\
C_2(\mu) &= \tfrac{1}{2}(1+2c)[3 - 2\mu + (2-4\mu)c].
\end{aligned}$$

(ii) Let the coefficients p_i, q_i *be such that the integral in the implicit solution (9) is reducible to the integral in the left-hand side of (19b), i.e.*

$$\frac{P_1(y)}{\sqrt{R_5(y)}} = \frac{y + c + 1/2}{w} \qquad \forall y \in \mathbb{R}.$$

Then the equation $(P_1(u)u')' + Q_3(u) = 0$ *has the following solution:*

$$\begin{aligned}
u(x) &= Z(v(x)) \\
v(x) &= 4\wp\left[\frac{\sqrt{r_5}}{p_1}(x + c_2); g_2, g_3\right] + \frac{1}{3}(\zeta_1 + \zeta_2 + \zeta_3)
\end{aligned} \tag{23}$$

where

$$g_2 = \tfrac{1}{12}[\zeta_1^2 + \zeta_2^2 + \zeta_3^2] \qquad g_3 = \tfrac{1}{432}[21\zeta_1\zeta_2\zeta_3 + 2(\zeta_1^3 + \zeta_2^3 + \zeta_3^3)]$$

and the function $Z(\mu)$ *is defined as a root of the equation* $Z^3 + B_2(\mu)Z^2 + B_1 Z + B_0(\mu) = 0$ *with the coefficients:*

$$\begin{aligned}
B_0(\mu) &= \tfrac{1}{8}[1 + 2\mu - 4\xi_0 + 8(2-\xi_0)c + 4(1-2\xi_0)c^2] \\
B_1 &= \tfrac{3}{4} - \xi_0(1+2c) + c^2 \\
B_2(\mu) &= \tfrac{3}{2} - \mu.
\end{aligned}$$

Numbers $\zeta_1, \zeta_2, \zeta_3$ *are defined by the following equations:*

$$\begin{aligned}
&\zeta_1 + \zeta_2 + \zeta_3 = 4 + y_0 + (4y_0 - 2)c + (4y_0 - 2)c^2 \\
&\zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_2\zeta_3 = 3y_0 + 5\xi_0 + 2(7y_0 + 5\xi_0 - 3)c + 6(3y_0 - 2)c^2 + 4y_0c^3 + c^4 \\
&\zeta_1\zeta_2\zeta_3 = -4\{y_0(4\xi_0 - 2) + 2\xi_0 + 2[9 + 2y_0(6\xi_0 - 1) - 2\xi_0]c + [3y_0(16\xi_0 + 1) \\
&\qquad -31\xi_0]c^2 + 8[1 + (4y_0 - 2)\xi_0]c^3 + (12 - 11y_0)c^4 + 2c^5\}
\end{aligned}$$

for arbitrary y_0, c *and* ξ_0.

We note that all square roots in the last propositions are arithmetic. All expressions in the roots are supposed to be real and positive for simplicity, such that all the solutions listed above are real single-valued functions. Due to lack of space we do not study the problem of derivation of coefficients p_i, q_i providing such assumptions are valid.

Reductions in the case $s = 1, n = 4$ ($g = 2$) can be found in the paper by Bolza [23], see also [24]. The corresponding formulae for exact solutions seem to be complicated for further applications.

3. Periodic solutions to the problem with linear diffusion and cubic adsorption functions

For application to adsorption–diffusion process modelling, it is of interest to consider the case $s = 1$, $n = 3$ in more detail, that is

$$(1 + ku)u'' + k(u')^2 + u(a + lu + mu^2) = 0 \tag{24}$$

where k, a, l, m are constant.

Condition (16) is now written as

$$c_1 = \frac{m}{5k^2}\left(\frac{1}{2k^2} + \frac{5a}{3m} - \frac{5l}{6km}\right)$$

which allows us to apply proposition 2 and to obtain the following solution to equation (24):

$$\begin{aligned} u(x) &= \frac{1}{4k} - \frac{5l}{12m} - \frac{10k}{m}\wp(x + c; g_2, g_3) \\ g_2 &= \frac{2klm + 5k^2l^2 - 3m^2 - 16amk^2}{240k^4} \\ g_3 &= \frac{25k^3l^3 + 15mk^2l^2 + 27klm^2 - 120almk^3 - 72ak^2m^2 - 27m^3}{43\,200k^6} \end{aligned} \tag{25}$$

where c is an arbitrary constant. This was first obtained in [26] using another approach.

A remarkable feature of the solution (25) is that it was found under minimal restrictions on the parameters of equation (24): namely, one had to determine only a value of an arbitrary c_1. Thus, this solution is valid for arbitrary coefficients in the corresponding equation and depends on one arbitrary constant c. For any other equation of the form (5) with $s > 0$ and $n > 2$ one cannot find a periodic solution (by means of inversion of the corresponding integral in (9)) without additional restrictions to the coefficients of $P_s(u)$ and $Q_n(u)$.

4. Bounded solutions

For applications, the most interesting solutions to equation (24) are among the explicit bounded periodic and localized functions. To find some of them we use the solution (25) and describe all cases of degeneration of the $\wp$-function into bounded functions with appropriate parameters c , g_2, g_3.

Let 2ω, $2\omega'$ be the primitive periods of the $\wp$-function defined by the condition $\mathrm{Im}\,(\omega'/\omega) > 0$, and let $\omega_1 = \omega$, $\omega_2 = \omega + \omega'$, $\omega_3 = \omega'$; $e_\alpha = \wp(\omega_\alpha)$, $\alpha = 1, 2, 3$ be roots of the equation $4e_\alpha^3 - g_2e_\alpha - g_3 = 0$, and $\Delta = g_2^3 - 27g_3^2$ be the discriminant [27].

In order to obtain the bounded solutions it is convenient to use the summation theorem [27] in the form:

$$\wp(z + \omega_\alpha) = e_\alpha + \frac{(e_\alpha - e_\beta)(e_\alpha - e_\gamma)}{\wp(z) - e_\alpha} \tag{26}$$

and relationships for the $\wp$-function and the Jacobi elliptic functions

$$\begin{aligned} \mathrm{sn}^2\left(\sqrt{e_1 - e_3}z, M\right) &= \frac{e_1 - e_3}{\wp(z) - e_3} \\ \mathrm{cn}^2\left(\sqrt{e_1 - e_3}z, M\right) &= 1 - \mathrm{sn}^2\left(\sqrt{e_1 - e_3}z, M\right) = \frac{\wp(z) - e_1}{\wp(z) - e_3} \end{aligned} \tag{27}$$

where $\{\alpha, \beta, \gamma\}$ is any permutation of numbers $\{1, 2, 3\}$, and $M = (e_2 - e_3)/(e_1 - e_3)$ is the modulus of the Jacobi elliptic functions.

It is well known that the behaviour of the $\wp$-function strictly depends on a sign of Δ.

4.1. The case $\Delta > 0$

In this case a pair of the primitive periods 2ω, $2\omega'$ exists, such that ω is real and ω' is pure imaginary. Then $\wp(z)$ is real only on the complex z-plane lines corresponding to the lattice of periods

$$\mathrm{Re}\,(z) = 2p\omega \qquad \mathrm{iIm}\,(z) = 2q\omega' \qquad p, q = \text{integer} \tag{28}$$

and on the lines corresponding to the lattice of half-periods

$$\mathrm{Re}\,(z) = (2p+1)\omega \quad \mathrm{iIm}\,(z) = (2q+1)\omega' \qquad p, q = \text{integer}. \tag{29}$$

The $\wp$-function has discontinuities on the real axis and, hence, on the lines obtained by means of shifting of it along the lattice of periods, i.e. on the lines (28), whereas it is bounded on the lines (29). Therefore, one can obtain a real bounded periodic solution $u(x)$ from (25) only if the arbitrary constant c shifts the real line of the variable x to the line $x + c$, which coincides with the one of (29). Thus $c = c_0 + (2q+1)\omega'$ with an arbitrary real c_0 and integer q. Furthermore, obviously $\wp(x + (2q+1)\omega') = \wp(x + \omega')$ for $\forall x$ and integer p. Hence, without a loss of generality one can take $c = c_0 + \omega'$, where c_0 is a real number and ω' is pure imaginary. Such c provides the shift of the real axis of the variable x to the line $\mathrm{iIm}\,(x + c) = \omega'$ of the complex variable $x + c$. Let us calculate the $\wp$-function value with an argument on that line.

When $\Delta > 0$ all $e_\alpha, \alpha = 1, 2, 3$ are real, $e_1 > e_2 > e_3, e_1 > 0, e_3 < 0$, and from (25)–(27) we obtain the following bounded periodic solution to (24) in the form of the 'cnoidal wave':

$$u(x) = \frac{1}{4k} - \frac{5l}{12m} + e_2\frac{10k}{m} + (e_3 - e_2)\frac{10k}{m}cn^2\left(\sqrt{e_1 - e_3}(x + c_0), M\right). \tag{30}$$

4.2. Case $\Delta < 0$

In this case the complex conjugated primitive periods 2ω, $2\omega'$ exist, and provide the transformation of fundamental parallelogram to a rhombus. Then $\wp(z)$ is real only on its diagonals, i.e. on the lines

$$\mathrm{Re}\,(z) = p(\omega + \omega') \qquad \mathrm{iIm}\,(z) = q(\omega - \omega') \qquad p, q = \text{integer}.$$

Because of $\omega + \omega' \in \mathbb{R}$, the function $\wp(x + \omega + \omega')$ is not bounded for real x. Furthermore, the simple transformations $\wp(x + \omega - \omega') = \wp((x + \omega + \omega') - 2\omega') = \wp(x + \omega + \omega')$ mean that $\wp(x + \omega - \omega')$ is not bounded, too. Therefore, in the case $\Delta < 0$ there are no bounded real solutions $u(x)$ in the form (25).

4.3. Case $\Delta = 0$

In this case one of two periods is infinite: $\omega = \infty$ or $\omega' = \mathrm{i}\infty$ (whilst the case $\omega = -\mathrm{i}\omega' = \infty$ is trivial).

The case $\omega = \infty$ corresponds to $e_1 = e_2 \neq e_3$. Writing $e_1 = e_2 = h$, we have $e_3 = -2h$ and

$$g_2 = 12h^2 \qquad g_3 = -8h^3 \qquad \omega = \infty \qquad \omega' = \frac{\pi\mathrm{i}}{\sqrt{12h}}.$$

Taking $c = c_0 + \omega'$, $\forall c_0 \in \mathbb{R}$ as above and using similar transformations, we obtain the cnoidal solution

$$u(x) = \frac{1}{4k} - \frac{5l}{12m} - h\frac{10k}{m} + \frac{30kh}{m}\mathrm{cn}^2\left(\sqrt{3h}(x + c_0), M\right).$$

Now $M = (e_2 - e_3)/(e_1 - e_3) = 1$, and, due to the relationship $\mathrm{cn}\,(u, 1) = 1/\cosh(u)$, the function $u(x)$ has the form of an *autosoliton* solution:

$$u(x) = \frac{1}{4k} - \frac{5l}{12m} - h\frac{10k}{m} + \frac{30kh}{m}\cosh^{-2}\left(\sqrt{3h}(x + c_0)\right) \tag{31}$$

i.e. $u(x)$ is bounded and differs significantly from a constant value only in the vicinity of the value $x = -c_0$ of real axis.

The case $\omega' = \mathrm{i}\infty$ corresponds to $e_1 \neq e_2 = e_3$. Then, writing $e_2 = e_3 = -h$, we have

$$e_1 = 2h \qquad g_2 = 12h^2 \qquad g_3 = 8h^3 \qquad \omega = \frac{\pi}{\sqrt{12h}}.$$

Now $M = 0$, and it follows from (27)

$$\wp(z) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2\left(\sqrt{e_1 - e_3}z, 0\right)} \equiv -h + \frac{3h}{\sin^2\left(\sqrt{3h}z\right)}.$$

These functions are bounded only if $\sqrt{3h}z = \pi/2 + \mathrm{i}w$, $w \in \mathbb{R} \Leftrightarrow z = \omega + \mathrm{i}w/\sqrt{3h}$. It is easy to see that one cannot find an appropriate c such that $x + c = \omega + \mathrm{i}w/\sqrt{3h}$, where $x, w \in \mathbb{R}$. Therefore, in the case $\omega' = \mathrm{i}\infty$ there are no bounded solutions of the form (25).

5. On values of physical parameters for existence of bounded solutions

In the physical problem of thin films growth described in the introduction it is interesting to interprete the conditions $\Delta > 0$ and $\Delta = 0$ in terms of physical parameters b, θ_0, ϕ_A and ϕ_D.

Supposing that the equation (5) obtained from (2) has the form (24), one can easily find the dependence of coefficients k, a, l and m on the physical parameters. Then, using $\Delta = g_2^3 - 27g_3^2$ and the formulae for g_2 and g_3 from (25), we can find the following explicit expression for Δ in terms of physical parameters $b, \theta_0, \phi_A, \phi_D$:

$$\Delta = \frac{(\exp\{\phi_D\theta_0\}b\phi_A)^6(\phi_A\theta_0 - 3)^2 F_4(\phi_D)F_8(\phi_D)}{3\,732\,480\,000\theta_0^6(1 - \theta_0)^6\phi_D^{12}(\phi_D\theta_0 - 2)^{12}}. \tag{32}$$

Here $F_4(\phi_D)$ and $F_8(\phi_D)$ are given polynomials of fourth and eighth degree with respect to an unknown ϕ_D, correspondingly, with coefficients depending only on ϕ_A and θ_0.

The solutions (30) and (31) are obtained from (25) for parameters ϕ_A, ϕ_D, b depending on whether $\Delta > 0$ or $\Delta = 0$. It is obvious from (32) that the inverse problem of analytical derivation of the parameters $\{\phi_A, \phi_D, b\}$ provided $\Delta = 0$ or $\Delta > 0$ is hardly solvable. However, it can be shown that, due to a special factorization of $F_8(\phi_D) = S_4(\phi_D)H_4(\phi_D)$ (with polynomial S_4 and H_4) for any ϕ_A and $b = \exp(-\phi_A/2)$ (this case corresponds to $\theta_0 = \frac{1}{2}$, which is very interesting for physics), all 12 roots ϕ_D of Δ can be found for any fixed ϕ_A and given value of b. Numerical analysis was made of the discriminant Δ as the function of the physical parameters based on the program, by which on the plane (ϕ_A, ϕ_D) for different b the following domains were separated:

(i) domain I, where bounded ($\Delta > 0$) and physically *meaningful* ($0 \leqslant \theta = \theta_0 + u \leqslant 1$) solutions of the form (25) exist;

(ii) domain II, where bounded ($\Delta > 0$) but physically *meaningless* solutions ($\theta < 0$, or $\theta > 1$) of the form (25), and three roots of equation (4) ($b_{\min}(\phi_A) < b < b_{\max}(\phi_A)$) exist, while the second root is taken as θ_0;

(iii) domain III, where bounded ($\Delta > 0$) but physically *meaningless* solutions ($\theta < 0$, or $\theta > 1$) of the form (25), and one root θ_0 of equation (4) ($b < b_{\min}(\phi_A)$, or $b > b_{\max}(\phi_A)$) exist.

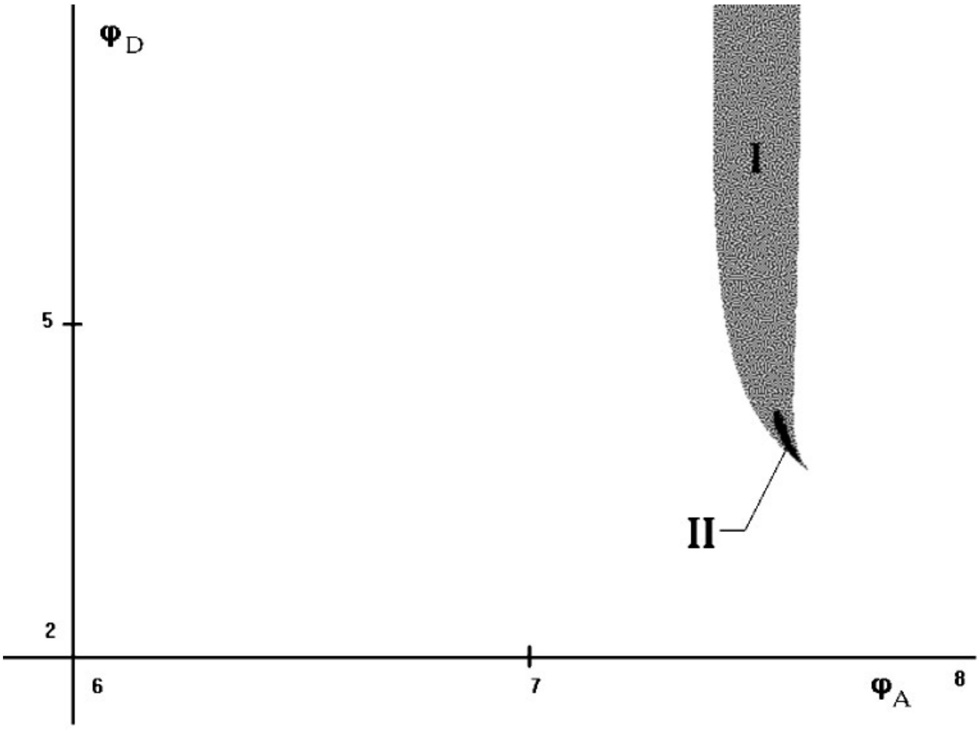

Figure 1. Domains of the (ϕ_A, ϕ_D) plane, where different types of solutions exist, for $b = 0.01$.

In figures 1 and 2 these domains are shown for some b, obtained as the result of calculations. In figures 3 and 4 the functions $\theta(x) = \theta_0 + u(x)$ are plotted, where $u(x)$ is defined by (30) or (31), and the number θ_0 is a solution of (4). The natural limits for the physical parameters are: $\phi_i = 4 \pm \varepsilon$, $|\varepsilon| \leqslant 4$, $(i \equiv A, D)$; $b > 0$, and $c_0 = 0$. The plots are shown for the values of parameters $\phi_A = 4.5$, $b = 0.1$ and various ϕ_D. For such b equation (4) has three roots, from which the second one was taken as θ_0 ($\theta_0 = 0.4$). The plots of $\theta(x)$ with values of ϕ_D providing $\Delta = \Delta(\theta_0, \phi_A, \phi_D) > 0$ are shown in figure 3, while the case of ϕ_D providing $\Delta = \Delta(\theta_0, \phi_A, \phi_D) = 0$ corresponds to figure 4.

It is seen from figures 1 and 2 that the most interesting domain I corresponds to physically meaningful values of b, ϕ_A and ϕ_D, which yield the stationary solution to the problem considered in either periodic (30) or localized (31) form.

6. Generalizations

In section 2 we described the method of finding some stationary solutions to equation (7) in terms of elliptic functions. We have noted that the implicit solution (8) for polynomial $A(u)$ and $D(u)$ reduces to the Abelian integral (9). A natural question arises: do reaction and diffusion functions exist, that are not polynomials and provide an implicit solution to equation (7) in terms of the Abelian integral? Let us show that the answer is positive.

In general, the Abelian integral has the form

$$I = \int R(u, v)\,\mathrm{d}u \tag{33}$$

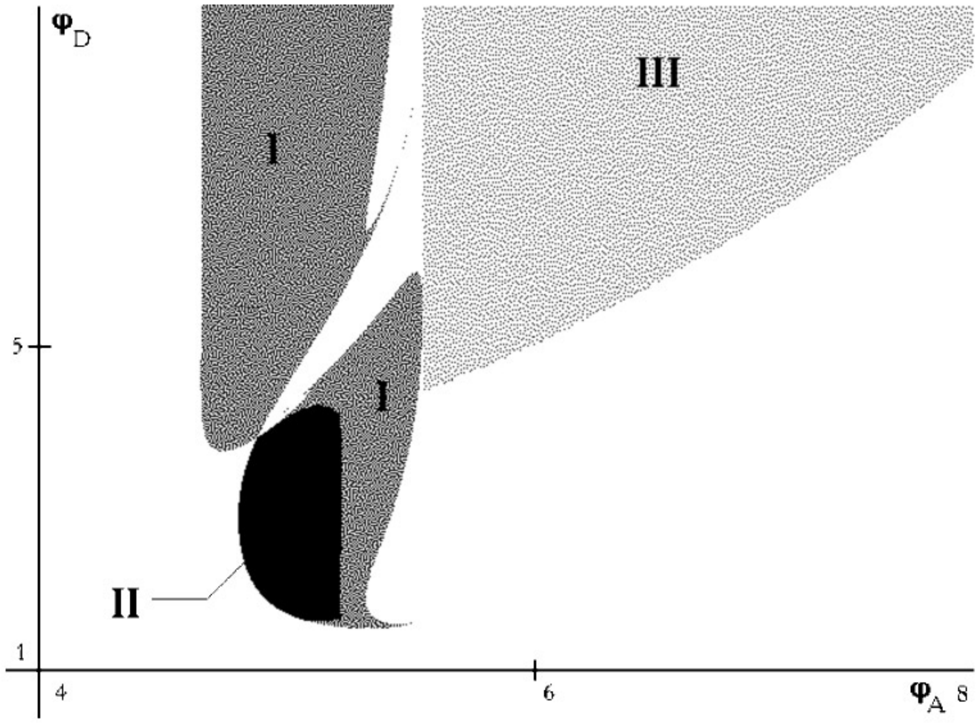

Figure 2. As figure 1, but for $b = 0.08$.

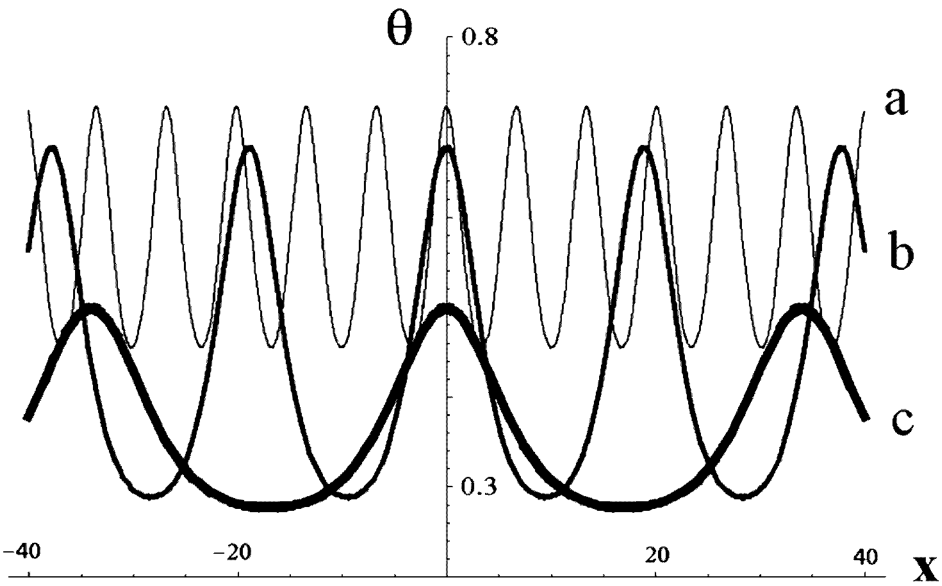

Figure 3. Periodic solutions $\theta(x) = \theta_0 + u(x)$ of the form (30) for $b = 0.1$, $\phi_A = 4.5$ and different values of ϕ_D: (*a*) $\phi_D = 4.1$; (*b*) $\phi_D = 3.2$; (*c*) $\phi_D = 1.8$.

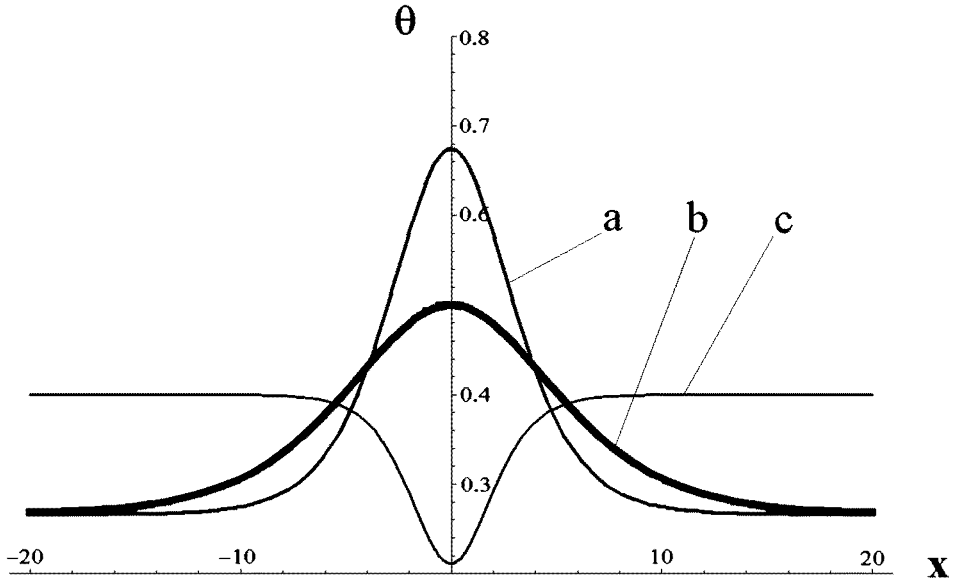

Figure 4. Autosoliton solutions $\theta(x) = \theta_0 + u(x)$ of the form (31) for $b = 0.1$, $\phi_A = 4.5$ and different values of ϕ_D: (*a*) $\phi_D = 3.1$; (*b*) $\phi_D = 1.8$; (*c*) $\phi_D = 5.6$.

where R is a rational function of its variables, while v is an algebraic function, i.e. a solution to the following equation:

$$P(u, v) = 0$$

with P being a polynomial of the two variables. The integral in the implicit solution (8) is Abelian if the following relationship is valid:

$$\frac{D(u)}{\sqrt{c_1 - 2\int A(u)D(u)\,\mathrm{d}u}} = R(u, v) \qquad \forall u \in \mathbb{R}$$

for some rational R and algebraic $v = v(u)$. Expanding this, we obtain the following relationship on functions A, D and R:

$$A(u)R^3(u, v(u)) + D'(u)R(u, v(u)) - D(u)\frac{\mathrm{d}R\,(u, v(u))}{\mathrm{d}u} = 0. \tag{34}$$

This can be interpreted from various points of view. If, according to physics, equation (7) contains the functions A and D satisfying (34) for some R, then the implicit solution arises in terms of the Abelian integral. On the other hand, we can consider this relationship as the equation on the function $w(u) = R(u, v(u))$ with coefficients defined by functions A and D. In this case, (34) represents the Abel equation for $w(u)$.

If A and D satisfy (34), then we can apply the approach proposed above. Namely, if (34) holds for $R(u, v) = 1/v = 1/\sqrt{P_{3,4}(u)}$ with $P_{3,4}$ being a polynomial of third or fourth order, then the inversion of the corresponding implicit solution yields the explicit solution in the form (17) or (15). If the hyperelliptic case occurs, that is, if $R(u, v) = (\alpha + \beta u)/\sqrt{P_5(u)}$, then propositions 3 and 4 can be applied, resulting in explicit solutions (21)–(23).

In particular, it follows from (34) that if $D(u)$ is *arbitrary* and $A(u)$ is defined as

$$A(u) = -(P_{3,4}(u)D'(u) + \tfrac{1}{2}P'_{3,4}(u)D(u))$$

for arbitrary polynomial $P_{3,4}$ of third or fourth order, then equation (7) has explicit solution of the form (17) or (15), respectively.

7. Discussion and conclusions

Firstly, we consider briefly the well known nonlinear heat conduction problem. The power functions $A(u)$ and $D(u)$ arise in equation (1) when used for modelling heat conduction and nonlinear combustion processes in dissipative media. These equations describe the spatial heat localization and blow-up effects (infinite growth of temperature during a finite interval of time): see, e.g., [7]. The corresponding stationary equation has the form

$$(u^\alpha u')' + u^\gamma = 0 \tag{35}$$

and (8) now yields the *general* implicit solution

$$x = c_2 + \frac{1}{\alpha+1}\int \frac{\mathrm{d}v}{\sqrt{c_1 - 2v^\delta/(\alpha+\gamma+1)}}$$

where $v = u^{\alpha+1}$, $\delta = 1 + \gamma/(\alpha+1)$. Thus one has the following possibilities:

(i) if $\gamma = \alpha + 1$ then $\delta = 2$, and the periodic solutions to equation (35) have the form

$$u = A_1 \sin^{1/(\alpha+1)}(A_2(x - c_2))$$

with arbitrary constants A_1 and A_2;

(ii) if $\gamma = 2(\alpha+1)$ or $\gamma = 3(\alpha+1)$ then, correspondingly, $\delta = 3$ or $\delta = 4$, and the periodic solutions in this case are found as:

$$u = B_1 \wp^{1/(\alpha+1)}(B_2(x - c_2); g_2, g_3)$$

with arbitrary constants $B_{1,2}$ and invariants $g_{2,3}$.

Thus, for every exponent α, γ corresponding to values $\delta = 3$ or $\delta = 4$ one can investigate the blow-up processes by means of extraction of a set of attraction for the corresponding unbounded stationary solutions (i) and (ii) from the set of initial functions for equation (1) with the power functions A and D.

Now let us briefly repeat the scheme proposed for derivation of the bounded solutions (30), (31) in a physical problem.

For given values of physical parameters ϕ_A, ϕ_D, b one has to calculate the invariants g_2 and g_3 and the solution $\theta = \theta_0 + u$ by means of (25). This solution is valid for any given parameters ϕ_A, ϕ_D, b and depends on the arbitrary constant c. In order to find out whether one can obtain the bounded periodic ('cnoidal') or the localized ('autosoliton') version of that solution by an appropriate choice of c one has to find the value of the discriminant $\Delta = g_2^3 - 27g_3^2$. If $\Delta < 0$, then the solution is unbounded for any c. If $\Delta > 0$, then taking $c = c_0 + \omega'$, where c_0 is an arbitrary real number and ω' is the pure imaginary half-period of the $\wp$-function with the invariants g_2, g_3, one obtains the periodic solution in the form (30), where e_1, e_2, e_3 are roots of $4e_\alpha^3 - g_2 e_\alpha - g_3 = 0$, descending by value: $e_1 > e_2 > e_3$. Finally, if $\Delta = 0$, then there are two possibilities for the roots e_1, e_2, e_3 (ordered as above): namely $e_1 = e_2 \neq e_3$ or $e_1 \neq e_2 = e_3$. The first case corresponds to an infinite real period of the $\wp$-function, and in this case one obtains the autosoliton solution of the form (31), where $h = e_1 = e_2$. In the second case the $\wp$-function has the infinite pure imaginary period, so the solution is unbounded for any c.

Finally, we should mention the experiments in thin film growth [18] which resulted in periodic nanostructures formation, often called 'quantum' wires and dots. Up to now there has only been a few solutions in closed form which could be useful to describe such complicated phenomena.

The solutions (30) and (31) may provide a reasonable description of the nanostructures observed.

Acknowledgments

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References

[1] Murray J D 1989 *Mathematical Biology* (Berlin: Springer)
[2] Gurtin M E and MacCamy R C 1977 *Math. Biosci.* **33** 35–49
[3] Newman W I 1980 *J. Theor. Biol.* **85** 325–34
[4] Al-Ghoul M and Eu B C 1996 *Physica* D **90** 119–53
[5] Ponce Dawson S, Lawniczak A and Kapral R 1994 *J. Chem. Phys.* **100** 5211
[6] Arrigo D J and Hill J M 1995 *Stud. Appl. Math.* **94** 21–39
[7] Samarsky A A, Galaktionov V A, Kurdjumov S P and Mikhailov A P 1987 *Blow-Up Regimes in Problems for Quasilinear Parabolic Equations* (Moscow: Nauka) (in Russian)
[8] Galaktionov V A 1990 *Diff. Int. Eqns* **3** 863–74
[9] Wilhelmsson H 1989 *Phys. Scr.* **39** 606–9
[10] King J R 1989 *Q. J. Mech. Appl. Math.* **42** 537–52
[11] Hill J M, Avagliano A J and Edwards M P 1992 *IMA J. Appl. Math.* **48** 283–304
[12] Dubrovskiy G V and Kozachek V V 1995 *J. Tech. Phys.* **65**(4) 124–41 (in Russian)
[13] Bogdanov A V, Dubrovskiy G V, Krutikov M P, Kulginov D V and Strelchenya V M 1995 *Interactions of Gases with Surfaces* (Berlin: Springer)
[14] Dubrovskiy V G, Cirlin G E, Kozachek V V and Mareev V V 1997 *Czech. J. Phys.* **47** 389–96
[15] Kreuzer H J 1991 *Phys. Rev.* B **44** 1232–9
[16] Brunauer S, Emmet P H and Teller E 1938 *J. Am. Chem. Soc.* **60** 309–19
[17] Fashinger W and Sitter H 1990 *J. Cryst. Growth* **99** 566–71
[18] Cirlin G E *et al* 1997 *Surf. Sci.* **377–9** 895–8
[19] Springer G 1957 *Introduction to Riemann Surfaces* (Massachusetts, PA: Addison-Wesley)
[20] Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1980 *Theory of Solitons: the Inverse Scattering Method* (Moscow: Nauka) (in Russian)
[21] Krazer A 1903 *Lehrbuch der Thetafunktionen* (Leipzig: Teubner)
[22] Burnside W 1892 *Proc. London Math. Soc.* **23** 173–85
[23] Bolza O 1887 *Math. Ann.* **28** 447–96 (in German)
[24] Belokolos E D and Enolskii V Z 1988 *Nonlinear Equations Integrable in Two-Dimensional Theta-Functions and Ember Manifolds* (Kiev: Preprint I.88 of IMP AS Ukranian SSR) (in Russian)
[25] Enolskii V Z and Salerno M 1996 *J. Phys. A: Math. Gen.* **17** L425–31
[26] Samsonov A M 1998 *Phys. Lett.* A **245** 527–36
[27] Bateman H and Erdelyi A 1955 *Higher Transcendental Functions 1–3* (New-York: McGraw-Hill)